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ABSTRACT

This teacher's guide to probability and statistics contains three major sections. The first section on elementary combinatorial principles includes activities, student problems, and suggested teaching procedures for the multiplication principle, permutations, and combinations. Section two develops an intuitive approach to probability through activities which study relationship of: intuitive, empirical and theoretical probabilities; simple and compound probabilities. Activities include work with chips, drawing cubes, coin tossing, drawing attribute blocks, and rolling dice. The final section presents an intuitive approach to statistics including activities with random events, random samples, analysis of a random system with two possible outcomes for each simple event. (JW)

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MATHEMATICS UNIT

PROBABILITY AND STATISTICS

SE

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Bernis Barnes (69, 70)

Carolyn Harris (69,70)

Phabhat Hazra (69)

Dorothy Hogan (69)

James Kirkpatrick (69)

V. P. Manglik (70)

S. S. Sachdev (70)

Frank Wyse (70)

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PREFACE

Man's most productive ways of predicting patterns, relationships and trends in our social and natural environments depend on his ability to handle data in a way that will reveal or yield them. In this unit we will discuss some of the methods that are used in the collection, organization, and analysis of data. This does not mean that you should rely totaly on these methods or that they are superior to other methods; but we feel that these, as well as other methods, should be investigated and evaluated so that there will be some understanding of their strengths and weaknesses.

This unit has three parts. Although the last two parts use some of the concepts covered in the first part, we feel that they can be successfully studied without studying it. However, we recommend that the parts be covered in the order they are arranged in this unit.

Most of the exercises can be changed by using different numbers on the faces of the dice, pennies, etc. This will give the teacher more flexibility in his classes.

We recommend that a copy of the following book be available for reference:

Spiegel, Murray R. Statistics. New York, New York: Schaum Publishing Co., 1961.

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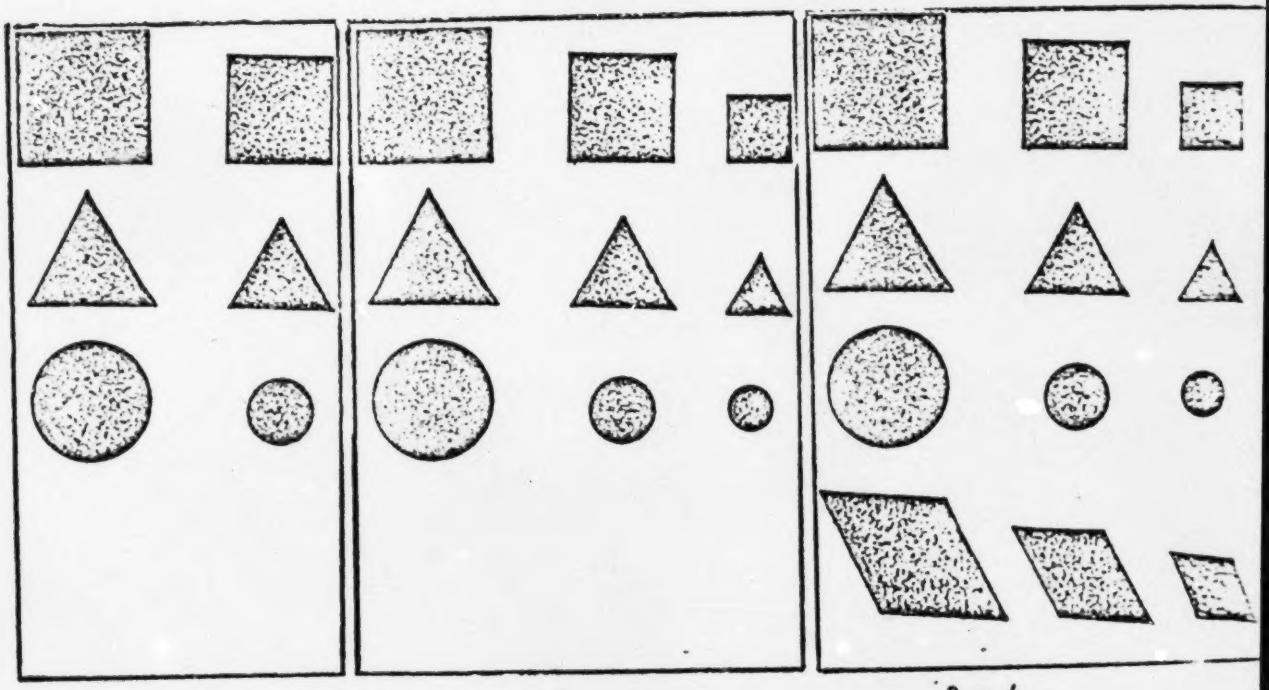
ELEMENTARY COMBINATORIAL PRINCIPLES

Materials: At least 4 sets of attribute blocks and 100 colored cubes. Also, use the attribute figures to make some cards with each card containing only figures of the same color, with each pair of cards of the same size containing different colors, and with the figures on each card arranged in a rectangular array as is indicated in the following:



2 x 3 3 x 3 3 x 4 etc.

I. The Multiplication Principle

Perhaps the most fundamental counting principle is the multiplication principle: If something is to be done in r steps and if there are n_1 ways of carrying out the first step, for each way of carrying out the first step there are n_2 ways of carrying out the second step, for each way of carrying out the first two steps there are n_3 ways of carrying out the third step, etc., then the whole sequence of r steps can be carried out in $n_1 \cdot n_2 \cdot n_3 \cdots$ n_r ways. Our first goal is to lead the students to appreciate and then

formulate this principle in some form, by considering examples.

A. The teacher may start this lesson by holding up one of the 2 x 3 cards.

1. Then focus the attention of the students on the attributes of the figures by asking:

   a. Which attributes are the same for all the figures on this card?

   b. Which attributes make the figures on the card different?

   c. Can you list all the attributes of these figures?

2. Establish a relationship between the different attributes and the number of figures.

   a. How many different sizes of attribute figures are on this card?

   b. How many different shapes of attribute figures are on this card?

   c. How many different attribute figures are on this card?

   d. What is the relationship between the number of sizes, shapes and figures?

3. Will this relationship hold for the 3 x 3 card? 3 x 4 card? etc.

B. Hold up 3 of the 3 x 4 cards and say:

1. Suppose you select these cards:

   a. How many different attribute figures are on each card?

   b. How many different attribute figures are on all three cards?

   c. How did you determine your answer?

2. Will this relationship hold for 4 cards? 5 cards? 6 cards? etc.

3. Generalizing the results.

   a. How can we determine the number of numbers to be multiplied?

   b. How can we determine the numbers to be multiplied?

   c. What generalization can you make?

C. Suppose we add to this set of attribute cards (holding up 4 of the 3 x 4 cards) a similar set of cards that are twice as thick as these, how many different attribute figures would be in both sets?

How did you determine your answer?

Related Problems

1. Draw some rectangular arrays made of squares and some parallepopiped arrays made of cubes, or use some square and cubic blocks to make rectangular or parallelopiped shapes respectively. Then give the number of squares or cubes in the array and show how you determined your answer in each case. (Make sure you try some square and cubic arrays.)
2. Select a set of five cubes of different color and put the numerals one through nine on the faces as specified below:
   a. If two cubes are used, how many two digit numerals can be represented when,
      (1) each of two digits are put on three faces of each cube?
      (2) each of three digits are put on two faces of each cube?
      (3) each of six digits are put one on each face of each cube?
      What generalization can you make about determining the number of two digit numerals when using two cubes?
   b. If three cubes are used, how many three digit numerals can be represented when,
      (1) each of two digits are put on three faces of each cube?
      (2) each of three digits are put on two faces of each cube?
      (3) each of six numerals are put one on each face of each cube?
      What generalization can you make about determining the number of numerals with n number of cubes?

3. You are going to walk six blocks through the streets of a city, but you do not care where you go. At each intersection you can either turn left or turn right or go straight ahead. How many different paths could you follow?

4. Consider the following results: $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, $5! = 120$, etc. Can you determine the rule that was used to get these results? (It should be brought out that we use this rule for positive integers only, except for $0! = 1$, which we do for convenience.) Also, consider for challenges such expressions as:

   (1) $\dfrac{n}{(n + 1)!}$

   (2) $\dfrac{(n + 2)!}{(n - 1)!}$, etc.

Materials: Poker chips and colored cubes.

II. Permutation

By using the multiplication principle, the students are to discover that the number of different sequences of r different things chosen from n is $n(n - 1)(n - 2) \ldots (n - (r-1))$ or $n!/(n - r)!$.

A. Start this lesson with another example of the multiplication principle. Put five of each of five different colors of poker chips in a box. Then say:

1. Select two chips one at a time from the box and remember the order in which they were drawn. How many different color sequences are possible?
2. Select three chips and ask the same question as above.
3. Select four chips and ask the same question as above.

Also ask:

What generalization can you make? Can you determine the number of color sequences that are possible in general?

B. This time put one of each of five colors in the box. Then say:

1. Select two chips one at a time with replacement (replacing the first before the second is drawn). How many different color sequences can be drawn?
2. Select three chips and ask the same question.
3. Select four chips and ask the same question.

Also ask:

How is this problem related to the previous problem?

C. Put one of each of five colors in the box. Then say:

1. Select two chips one at a time without replacement and remember the order in which they were drawn. How many different color sequences can be drawn?
2. Select three chips and ask the same question.
3. Select four chips and ask the same question.

Also ask:

What generalization can you make? Can you determine the number of color sequences that are possible in general?

Related Problems

1. The Greek alphabet has 24 letters. How many fraternity names 3 letters long can be formed?
   a) When all three letters maybe the same?
   b) When none of the letters is the same.
2. If 3 (then 4, 5, etc., n) cubes of different colors are placed in a line, how many different arrangements by color are possible?
3. In how many ways can 2 (then 3, 4, 5, etc., r) people choose seats in a room containing 2 chairs? 3 chairs? 4 chairs? 26 chairs? n chairs?

III. Combinations

By using the multiplication principle, the students are to discover that the number of groups of size r that can be selected from n things is determined by the expression $\frac{n!}{r!\ (n - r)!}$.

A. This lesson allows the students to count the number of combinations of points by counting the number of different polygons.

1. Number of lines -- combinations of two points.

   a) If 100 points are evenly spaced on the circumference of a circle, what is the maximum number of lines that can be drawn connecting any pairs of these points?

      For students that have difficulty figuring out this number, suggest that they use smaller numbers of points such as 2, 3, 4, etc., and then look for a pattern.

   b) How many lines would it be for n points?

2. Number of triangles — combinations of three points.

   a) If 100 points are evenly spaced on the circumference of a circle, what is the maximum number of triangles that can be drawn using the points as vertices?

   b) After counting the number of triangles for 3, 4, 5 points, write a general expression.

3. Number of quadrilaterals -- combination of four points.

4. Can you generalize this pattern for n points so it will work for any polygon with r vertices ($r < n$)?

B. To relate this lesson to the previous one, put five chips of different color in a box. Then say:

1. If you close your eyes while selecting two chips one at a time from the box and tossing them on a desk, how many different pairs of color are possible?

   What information is lost by drawing chips with your eyes closed? Does this affect your answer?

2. Suppose you close your eyes while selecting three chips from the box, how many different triples of color are possible?

After trying four or five of these ask:

What generalization can you make?

Related Problems

1. Suppose you are walking home from school and must go 7 blocks east and 8 blocks north, if you walk some time north and some time east, changing directions (at intersections) whenever you feel like it, how many different ways do you have to walk home?

2. Take three cubes of different colors and consider them to be people (girls: R, B, G). No matter how they are arranged in the room, they are the occupant of the room. Suppose the room looks like this

R	B	G

   a) Suppose one girl leaves the room, how would the possibilities look?

   b) Suppose two girls leave the room, how would the possibilities look?

   c) Suppose all three leave the room, how would the possibilities look?

   d) How many different possibilities?

I. Intuitive Probability. If the students already have an intuitive feeling for probability, the teacher may wish to omit this activity. But for other students, this exercise should prove to be valuable for understanding probability.

Materials: Put four numbered boxes on the desk with each containing the following known numbers of chips:

Box 1 has 10 yellow and 1 red chip.

Box 2 has 10 green and 2 red chips.

Box 3 has 15 white and 2 red chips.

Box 4 has 15 orange and 3 red chips.

Mix the chips in each box well. Then say:

A. If you draw a chip and get a red one, you will get a prize:

1. What are your chances of winning? Good? Fair? Terrible?
2. Are your chances better in one box than another? If so, which one?
3. Who can arrange the boxes according to his chances of winning?
4. What affect your chances of winning? Size of the box? Number of chips? Number of red chips? etc.
5. What is the most precise way to state your chances of winning?

B. Allow the students to test their predictions by drawing chips from the boxes.

II. It's In The Bag: This experiment is a study of the relationship of intuitive, empirical and theoretical probabilities.

Materials: Put 8 red, 6 blue, 4 yellow and 2 green cubes in a bag.

A. After mixing the cube blocks in the bag, ask the following:

1. What are the chances of selecting a red block from the bag?
2. What are the chances of selecting a red or green block?

etc.

In order to answer these questions, it is necessary to know answers to the following questions:

Are there any red blocks in the bag? If so, how many?

Are there any other colors in the bag? If so, what are they?

What is the distribution of colors in the bag?

What is the total number of blocks in the bag? etc.

3. How can we answer these questions without seeing all the blocks in the bag?

If the following is not suggested, you may say:

B. Suppose each student is permitted to select one block from the bag and replace it, can you determine which colors are in the bag?

Then pass the bag around and let each student select a block and replace it. Shake the blocks well after each draw. Usually, the students will keep records of the draws themselves. When everyone has selected a block, ask:

1. What colors are in the bag?
2. Can you rank the colors according to their frequency of occurence?

At this time, the students may want to pass the bag around again. After a second time around, the students should know which colors are in the bag and be able to rank them largest number to smallest number. Ask:

3. What is the distribution of colors in the bag?

   a. How much confidence do you have in your prediction?

   b. What would cause you to have more confidence in your prediction?

Once the students feel that their distribution of colors is correct, they should estimate the probability of some of the colors based on the data collected from the drawings. These estimated probabilities are also called empirical probabilities.

C. To find the empirical probability of each color, ask:

1. What is your chance of selecting a red block in one draw?

   a. How would you determine your chance of selecting a red block?

      (1) How many times was a red block drawn?

      (2) How many times was a block drawn?

      (3) What portion of the blocks drawn were red?

   b. Will this procedure work for other colors?

2. What is your chance of selecting a blue block in one draw?

* 3. What is your chance of selecting a red or blue block in one draw?

* 4. What is your chance of selecting a red block twice in two draws?

etc.

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* You may wish to deal with these questions in the next experiment.

D. When the students understand how to determine empirical probabilities, take the blocks out of the bag so they can find the corresponding theoretical probabilities and compare these results with the empirical results. Ask:

Can you determine the theoretical probability that corresponds to each empirical probability found in section C above?

1. What is the probability of a block being red on the first draw?
2. What is the probability of a block being blue on the first draw?

etc.

In order to make a good comparison, all fractions should be expressed with a common denominator.

Here is how one class went:

I (the teacher) placed on display a bag with a collection of colored cubes in it. (Other materials could easily have been used.) Questions such as the following were used to get started: What are the chances of selecting a red or green block from the bag? a red block? ... etc.

In order to answer these questions, it was necessary to know answers to the following: Are there any red blocks in the bag, and if so, how many? Are there any other colors in the bag? What is the distribution of colors? How can these questions be answered without looking into the bag? Suppose you were permitted to select one block at a time from the bag and replace it, do you think you could answer the question of what are the chances of selecting a red block on one draw? I passed the bag around so that each student might select a block and replace it and I shook up the blocks after each draw. (For this particular experiment, I had placed 8 red, 6 blue, 4 yellow and 2 green cube blocks in the bag.)

The first three blocks selected were red, but the fourth was blue. At this point students started to keep a record of the colors selected. On or about the tenth draw a green block turned up. It was no shut case as to the colors in the bag, and one's reward was to select a color different from what others were selecting. After everyone had selected a block, the initial question was asked again. Can you determine the distribution of colors now?

Most of the students wanted to run the experiment again before answering the questions. After the second time around with each student selecting and replacing one block and all of the students keeping a record of what happened, it was agreed that the distribution was ranked from the largest number to the smallest number, as follows: red, blue, and the same number of greens and yellows. Are you confident that this ranking is correct? "No," was the reply of some, "let's test it one more time." After this experiment the situation for the reds and blues was assured, but the greens and yellows were still in doubt. We did not try to settle the question of the greens and yellows at this time, but tried to learn if anyone had an answer to the chances of selecting a red block. (Everyone thought that the chances were good.) How many times was a red block picked? How many times was a block drawn from the bag? After everyone was satisfied that the ratio of these two results would answer the question, consideration was given to the question of selecting a blue block, a green block, a yellow block. The question of selecting a red or yellow, a red or green, etc., were considered and answered.

Then I told the students the number and the colors of the blocks in the bag and asked them to answer the same questions. When answers and explanations were given I asked for a comparison. In order to make a good comparison, they were asked to express each of the chances with a common denominator. Later we referred to the first answer as the empirical chances and the second as the theoretical chances. We also called the information about the blocks the sample space for the probabilities.

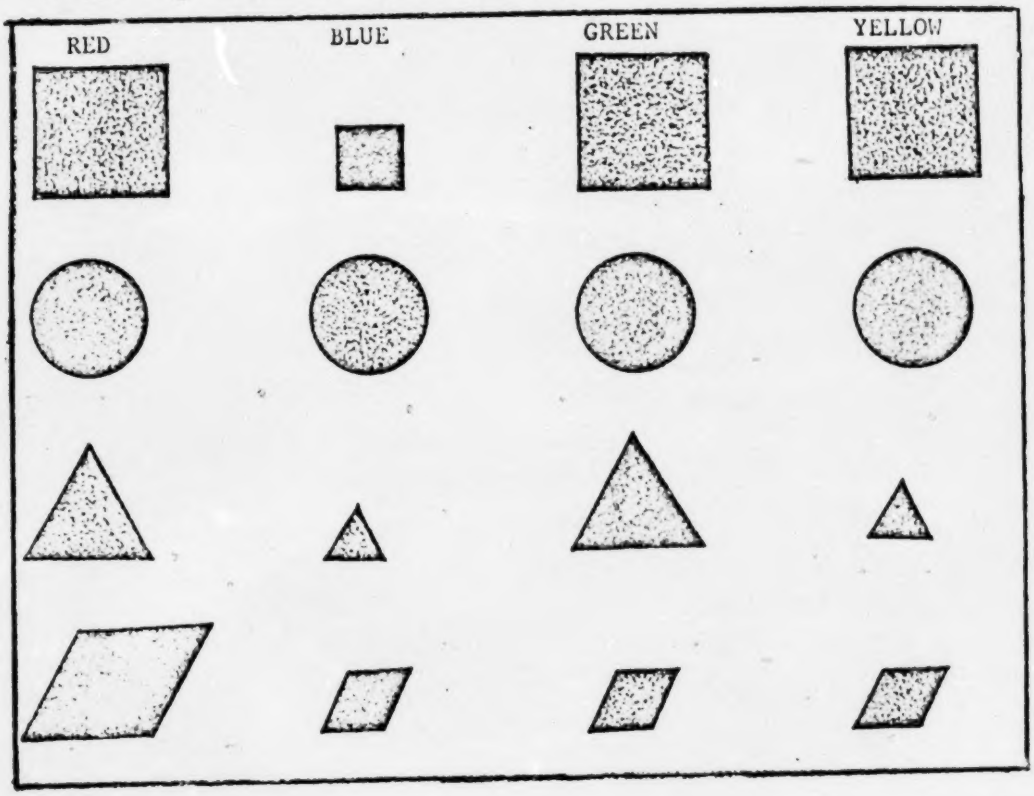
The next day the students conducted several experimants. After an answer was determined by experimentation, the students were asked to determine the theoretical probability.

Related Exercises:

1. Two coins: What is the empirical probability of tossing two coins and getting two heads?
2. One die: What is the empirical probability of getting a four in one toss?
3. Two dice: What is the empirical probability of getting a total of five in one toss?
4. A bag with red and green poker chips: What is the empirical probability of getting a green chip on the first draw?
5. A bag with blue, yellow, green, orange and red poker chips: What is the empirical probability of getting a blue chip on the first draw?

III. What's Expected on a Draw? This lesson is a study of the relationship of simple and compound probabilities.

Materials: A chart and at least 4 sets of attribute blocks which correspond to the figures of the following chart:



To make it easier to find the theoretical probabilities, display the chart containing the attribute stickers. For students that wish to get an intuitive feeling for these probabilities, allow them to work in groups with a set of the attribute blocks. As they finish one relationship, they should work on another. However, it may be necessary to ask questions such as those given in the following sections A - C. Start the class by saying:

A. Each sticker on the chart corresponds to one of the blocks in the bag.

Then draw a block from one of the bags and ask:

1. Can you determine the theoretical probability of selecting this block?

As other blocks are drawn, ask the same question if it is necessary.

You may also need to ask the following kinds of questions.

2. What is the probability of getting a given block on the first draw?

3. Is the probability the same for each block? If not, why not?

During this activity some students may not agree with the assumption that attributes are equally likely. Such concerns should be discussed and resolved. For example, if circular and square blocks are not equally likely, then what should be the probability of each?

B. What are some general relationships of the probabilities of combinations of attributes?

1. Attributes that are independent--the occurence of one does not affect the occurence of the other. Say:

a. If I draw a block from the bag,

(1) What is the probability of it being a red block?

(2) Waht is the probability of it being a square block?

(3) What is the probability of it being a red and square block?

b. Are these three probabilities related? If so, how?

c. Will this relation always hold?

(1) Will it hold for blue and circular blocks?

(2) Will it hold for circular and square blocks? etc.

d. Can you predict when this relation will hold?

e. Give some other examples where this relation holds.

f. Can you state what the relationship of the probabilities is?

2. Attributes that are mutually exclusive--the occurence of one excludes the occurence of the other. Say:

   a. If I draw a block from the bag,

      (1) What is the probability of it being a red block?

      (2) What is the probability of it being a blue block?

      (3) What is the probability of it being a red or blue block?

   b. Are these three probabilities related? If so, how?

   c. Will this relation always hold?

      (1) Will it hold for circles or squares?

      (2) Will it hold for red or square?

         (a) What is the difference in the probabilities?

         (b) Can you state a rule for such cases? etc.

   d. Can you predict when this relation will hold?

   e. Give some other examples where this relation holds.

   f. Can you state what the relationships of the probabilities are?

3. If the probabilities of simple attributes are used to determine the probabilities of compound attributes, when should you add and when should you multiply?

C. What are some general relationships of the probabilities of successive events?

Conditional probability-- the probability that one event occures given that another event has occured.

1. Successive draws that are independent events-- drawing with replacement. Draw two blocks one at a time and replace each block after the outcome is recorded on each draw. Then ask:

   a. What was the probability of getting that block on the first draw?

   b. What was the probability of getting the second block on the second draw?

      (1) Is the probability the same for both draws?

(2) Did the drawing of the first affect the chance of drawing the second?

c. What was the probability of drawing those two blocks on the first two draws? How many different pairs of blocks are possible? (You may need to use tree diagrams to determine the number of pairs.)

d. Are the three probabilities related? If so, how?

e. Will this relation always hold?

(1) Will it hold if the first draw is red and the second draw is blue?

(2) Will it hold if I select the second without replacing the first?

f. Can you predict when this relation will hold?

g. Can you state what the relationship of the probabilities is?

2. Successive draws that are dependent events--the occurence of the first affects the occurence of the second.

Draw two blocks without replacing the first before drawing the second. Then ask the same questions as above.

Related Problem

Which of the following events are independent and which are dependent?

a. Each die in the rolling of a pair of dice?

b. Each toss of the tossing of a coin twice?

c. Each draw of the drawing of a pair of cards from a bridge deck with replacement?

d. Each draw of the drawing from two of the four boxes in lesson I, "Intuitive Probability"?

e. Each coin in the tossing of two coins?

f. Each draw of the drawing of a pair of cards from a bridge deck without replacement?

g. Each height of the heights of two persons? What if they are father and son?

Materials: Colored Cubes.

IV. Crazy Dice. I suspect that most students will have a rough idea about the numbers that will be obtained in the tossing of two standard dice. They might know from experience that 7 is easiest to get and 2 or 12 is pretty unlikely to turn up. This lesson will attempt to analyze some of these ideas and compare the empirical and theoretical results.

In this lesson you are to look at another random system, two non-standard dice of different color--red and yellow. First, have each student to write the numbers 1,2,3,7,8, and 9 on the faces of the dice. Then each student makes 20 rolls of the dice and records the results. In recording the data, most students will probably list sequentially the sum of numbers on the upward face of the dice after each roll.

A. After each student has rolled the dice 20 times, start the analysis of the empirical results by asking qualitative type questions such as:

1. Is it reasonable to find that more nines than tens were rolled? Would this mean that the dice are loaded?
2. Should there have been some rolls of 16? 17? 19?
3. What number has the greatest chance of turning up on a single roll?
   a. Is the chance of getting a 9 on the next roll more or less than the chance of getting a 10? 8 or 16? etc.
   b. What number has the greatest chance of turning up on a single roll?
4. Suppose we look at the data of the entire class:
   a. What would be a good way to organize it?
   b. What predictions can we make based on the data?
      (1) Should the outcomes be symmetric? If so, about which number?
      (2) How many times did the class get a sum of 10? 9? etc.
      (3) How many different sums did we get?
      (4) What is the probability of getting a sum of 10? 11? etc.
      (5. What predictions can you make based on the data?
   c. Can you draw a probability curve for this system?

B. In this section, analyze the expected (theoretical) results for this experimant.

1. If one position is allowed for each different way of getting a sum, how many positions are needed for this experiment?

   a. How many different sums should we get?

   b. How many different ways can we get a sum of 4? 8? 11? etc.

   c. In how many different ways can we get sums?

2. What would be the probability of each position?

   What is the probability of a R7-Y3? R9-Y7? etc.

3. What is the probability of getting a sum of 4? 9? 10? etc.

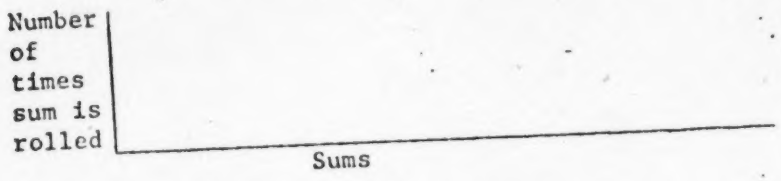
4. Can you draw a theoretical probability curve for this system?

   (Encourage the students to put the different sums in ascending order for the curve.)

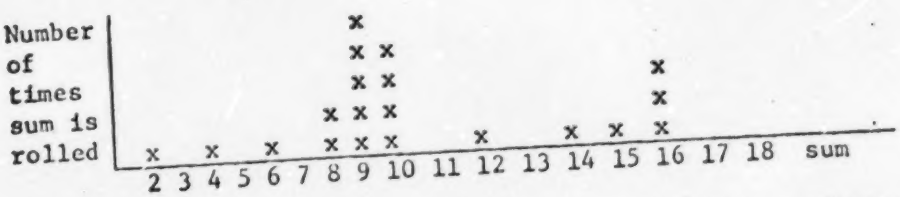
C. Compare and discuss the two curves in sections A and B; especially, their shapes.

The Way one Class Went:

Let's look at another random system, a pair of dice with the numbers 1,2,3,7,8,9 on their faces. Roll these dice 20 times and record your results. How can the results be plotted? Something of the following form was suggested by the class.

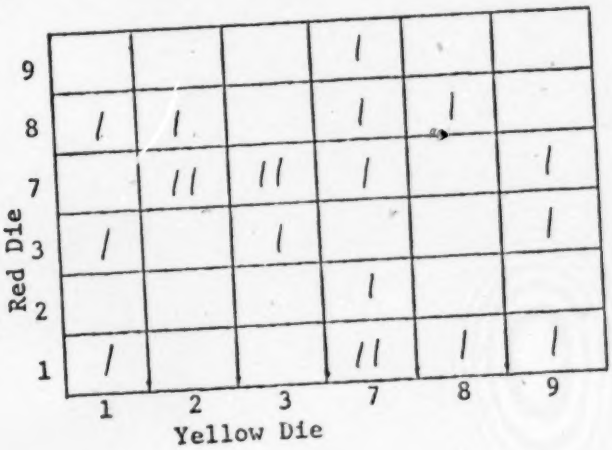


How far out shall we make the axis of our graph? What is the smallest number that can be rolled? What number do you think has the greatest chance, or probability, of being rolled. What number has the next greatest probability of being rolled? 3rd greatest? What numbers have the same probability of being rolled? If we use the rolls of the entire class, how many of the rolls give 10? 16? 20? 7? (Both of these last two numbers have zero probability of being rolled.) Finally, we came up with the following graph.



In the above graph, we marked an x for each time a given number was rolled. But we could get most numbers in more than one way. For example, we could roll a 10 by having a 3 on the red die and a 7 on the yellow die, or a 7 on the red and 3 on the yellow. If we want to keep track of what each individual die does on a roll, how might we plot our results?

If something like the following method had not been suggested by the students, it would have been suggested by the teacher. The relative merits of both methods were discussed by the class.



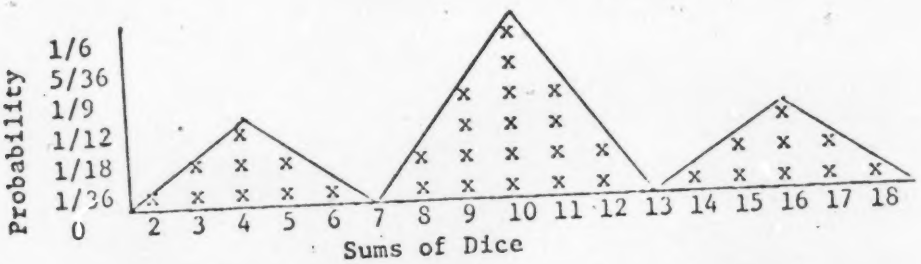
What is the probability of rolling a R1-Y7 relative to a R7-Y1: that is, what is the probability of getting a point in this square versus that square? What is the probability of getting a point in the R7-Y7 square versus the R3-Y2 square? A few more, then, what is the probability of rolling a R3-Y3? R2-Y7? R3-Y2,or R7-Y1? etc. That is, what is the probability of landing in either of these two squares? These 4 squares? These squares (half the board)?

All right, using this square lattice (the one above), can you tell me the probability of rolling a 2? 4? 10? How might we label the lattice points in order to find the probability of rolling any number?

Red Die \ Yellow Die	1	2	3	7	8	9
9	10	11	12	16	17	18
8	9	10	11	15	16	17
7	8	9	10	14	15	16
3	4	5	6	10	11	12
2	3	4	5	9	10	11
1	2	3	4	8	9	10

Sum Matrix

Let's make a graph of the probability of getting any given number. How should we label the ordinate? How does it compare with our earlier guess?



Related Exercises

1. Analyze a pair of standard dice and compare their theoretical probability curve with that of the 1,2,3,7,8,9 dice. Can you predict what the curve will look like before doing the problem?
2. Analyze a pair of dice with the numbers 4,5,6,7,8,9 on their faces. What will the theoretical probability curve look like?
3. Analyze the system of a penny and a nickel. Can you draw a probability curve for this system?

Materials: Colored cubes.

V. What's On The Dice? To start this lesson, give each student a copy of the data on the next page. Then say:

After numbers were put on the faces of two dice, they were rolled 400 times. You have a copy of the results of those rolls. Can you determine from this information which numbers were on the faces of the two dice?

A. I would not provide the students with a systematic way to handle these data, or any way to discover the numbers written on the faces of the dice. First, let us see what they can do. Even if they never do it "our way," it may be -- and it has already happened -- that they will invent a better way than anything we have thought of. Besides, even if their scheme is not very satisfactory, it is their scheme, and they ought to see what it can do, and what it cannot do; only then will they be able to appreciate the better schemes that other people have.

However, if they cannot get started after several attempts, the following questions might be useful to ask students, either individually or as a group, to help them to find systematic ways to proceed:

Are there any strategies for figuring out what was written on the dice or must you proceed by a guess-and-check method?

a. Are there particular numbers, or patterns of numbers, which give clues? Perhaps the lowest sum, or the highest sum? Perhaps there are other numbers or sequences of numbers?

b. Do you think you have all the possible different sums? What is the maximum number of different sums that are possible?

(Note: Sometimes the same sum can appear as a sum of two numbers in different ways: $2 + 5 = 7$ and $1 + 6 = 7$.)

c. As we accumulate a large amount of information about the sums that turned up, are there any peculiarities of the data that tell you anything? Since some sums appear much more often than others, does that mean anything?

d. Can we determine a sample space for this experiment?

e. Can you take a complete set of 36 sums (some doubled, etc.) and arrange them in only one 6 by 6 chart, (except for mirror images)?

The following numbers were the sums of the numbers on the upward faces of the two dice after each roll.

7	18	7	26	69	12	12	57	16	69	26	11	7	11	54	69
22	8	11	18	10	31	11	22	16	33	27	37	18	12	33	16
54	33	18	22	35	39	31	19	7	7	21	20	16	82	69	22
39	54	44	69	22	7	26	27	65	8	16	41	20	10	21	22
65	20	8	14	16	54	7	10	7	18	37	16	18	10	29	20
57	69	8	29	57	35	29	39	31	39	54	65	5	7	5	37
37	7	59	33	44	12	82	35	5	33	57	33	59	5	19	22
16	5	41	31	8	31	33	16	26	20	33	33	33	18	20	22
4	31	14	35	16	16	26	12	41	39	8	33	41	20	29	82
22	54	39	12	26	10	54	10	69	35	82	44	54	35	16	57
22	10	10	33	82	33	39	10	20	22	16	18	54	5	5	10
26	39	16	10	29	12	37	5	10	57	57	44	22	14	26	29
39	5	54	29	65	16	31	19	27	54	82	26	21	82	18	82
26	33	21	57	5	44	37	16	21	11	41	31	21	16	8	21
82	20	41	16	19	44	16	31	7	41	26	26	31	10	5	65
31	8	54	16	8	5	7	65	10	65	44	59	26	33	41	18
10	10	69	59	22	7	16	54	59	37	27	31	19	69	10	57
10	26	82	82	14	12	54	21	65	19	59	41	37	82	59	69
29	14	5	8	37	57	22	5	11	26	10	12	41	19	35	16
12	16	37	21	33	65	44	12	57	10	16	26	21	69	31	39
16	12	29	54	41	82	8	54	21	18	19	37	41	44	29	22
16	54	82	54	11	5	16	26	26	20	16	54	16	22	20	10
54	22	7	20	31	19	21	5	33	29	35	27	26	16	7	82
57	35	39	12	29	21	26	19	82	54	8	7	20	26	11	20
18	16	31	18	44	10	44	7	22	7	54	31	44	5	8	18

What Some Students Did:

1. Some recorded the different sums that came up, but did not keep track of the number of times each sum occured. Since it is important for students to discover for themselves what is important in a real situation, and what is not, I allowed them to go ahead and do what they could, without the frequencies. Eventually, they saw from what others, or saw for themselves, that they needed this data. Then, and only then, did it make sense to collect it.

a. Even when the frequencies are recorded, they "do not always tell the truth," as one person said. This makes sense, and again, it is important for students to learn the difference between theoretical and actual distributions of results. After 360 tossings of the dice, a sum that occured as only one combination of the dice came up more often than a sum that was the result of two different combinations. So, when do you have enough data? Do you ever? How many sums do you need before you can say what the real sums are?

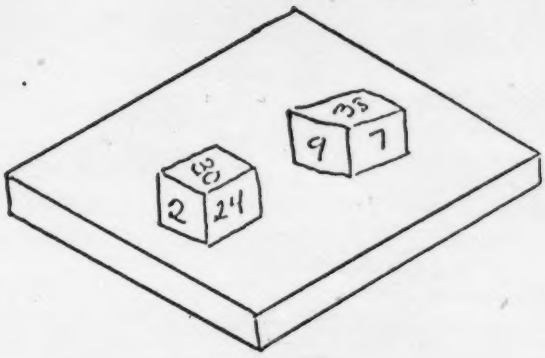
b. Many students, as soon as they hear that "22" has occured as a sum, will make a guess as to what combination produced it: "Let's say 20 for a face on the first die, and 2 for the second die." This process bogged down after the seventh or eight sum was tossed. Why? Is there some variation of that process which would be feasible?

2. Some students looked for patterns in the list of sums. The lowest number, 5, and the highest, 82, stand out. The gap between the highest, and the next highest, 69, intrigued some students. Others wanted to begin with the smallest number, 5, because "there are fewer combinations that add up to five than the other sums." Both reasons are sound; they both lead to productive investigations; yet they are, in a sense, contradictory. Here are two completely opposed starting points that can both lead to the solution. Is that not important for students to find out?

a. Some people took bare cubes and made up their own dice, like the ones that were used, and made all kinds of charts and diagrams to see how these dice work. The usual chart involved labelling six columns with the numbers written on the faces of one die, and six rows with the numbers written on the faces of the second die; then, the positions in a certain row under a certain column were marked with the sum of the two face numbers. (See figure 1.) They did not know what the faces were so they made up their own dice to see what pattern would develop. If the numbers going to the right, or going down, are in increasing order, then a curious and important pattern occurs. Let us show this by an example, which is not the dice we were working with.

Numbers on faces of second die	Numbers on faces of first die: 5	7	19	25	38	90
2	7	9	21	27	40	92
11	16	18	30	36	49	101
15	20	22	34	40	53	105
24	29	31	43	49	62	114
26	31	33	45	51	64	116
30	35	37	49	55	68	120

figure 1



The sum here is 68.

The first thing you notice is that you do not get 36 different sums, in fact, 31, 40 and 49 repeat. One might try rolling that pair of dice to see how long it takes before: (1) you get all the predicted sums, and (2) it becomes obvious which sums are supposed to repeat twice.

More important, for some students, is that there is a specific pattern to the numbers, providing that the column and row headings are put in increasing order. (A completely arbitrary decision, since the faces of a cube are not in any order at all.) One notices that the difference between adjacent numbers, like 7 and 9 in the top row, is maintained in the second (16 and 18) row, third (20 and 22) row, and so on down. The difference between 36 and 49, in the second row is repeated in the first row (27 and 40), in the third row (40 and 53), and all the way down. The difference between 9 and 40 in the first row is repeated all the way down: (18 and 49; 22 and 53; etc.). One can say the same thing about differences of numbers in the same column, 7 and 16 in the first column, 9 and 18 in the second column, etc. What is the general proposition here? What are its implications?

b. One person tried to work out his chart by putting the lowest and highest sums in the two obvious corners. Then he decided that 69, the next highest number, would have to sit either right above or immediately to the left of 82. (Do you agree?) He put it above 82, arbitrarily. (See figure 2.) In order to produce a "consistent chart" he had to find two sums that differed by 13 for the two positions to the left of those already fitted in. He proceeded by working down the list from the highest numbers, so that he asked about 65; where whould it be? If it is to be placed to the left of 82, (see figure 3), then that meant that above 65 should be a number 13 less than 65. But 52 is not on the list. He was forced to put 65 above the 69 in the last column. (Was he forced?) He tried the next number down, 59. Where does it go? Either above the 65, or to the immediate left of 82. (See figure 4.) (Really?) But then

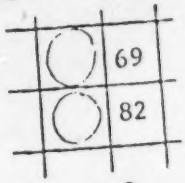


figure 2

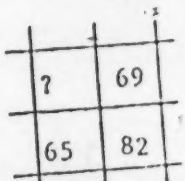


figure 3

what about the number above it; it would have to be 13 less than 59, or 46, which is not on the list. So 59 goes to the top of the last column. (Why?) Proceeding in this fashion, he produced, very soon, a "consistent chart."

	65
?	69
59	82

figure 4

c. Someone else began with the smallest number, 5, in the upper left hand corner, and asked: where does 7 go? He arbitrarily put it to the right of 5, since it had to be either there or directly under 5. (Really?) Then he asked: what about 8? Where does it go? He proceeded up from the smallest number step by step. He,too,got a consistent chart. It was almost the same as the one obtained by the person who began with the largest numbers. It was in fact, the mirror image, where the rows were exchanged for the columns, and vice versa. The same mirror image can be obtained by exchanging the roles of the two dice. And there is no reason why any particular die should be on top or on the side.

B. Axiomatics

At this point, it is worth taking time to notice that we have abstracted a miniature axiomatic system, from the context of the problem; we are not interested in what the numbers are on the twelve faces. Say:

1. In this system, we have focussed on a certain kind of square chart, with 6 columns and 6 rows, and with the special properties that:

   a. The numbers increase as you go down, and as you go to the right.

   b. If you pick two numbers in the same row and note their difference, you can go below those two numbers down or up to another row and find two other numbers with the same difference.

   c. Similarly, you might have begun with two numbers in the same column, and expected the difference to remain constant.

2. Suppose we had a square (6 by 6, or 9 by 9, it does not matter) filled with numbers, satisfying those three conditions; let's call that a "consistent chart." What properties can we deduce?

   a. Which properties are there in the charts we have made that would be a good way to begin our search?
   (Ans. The highest number appears at the lower right and the lowest number appears at the upper left.)

b. How about the next lowest number, what are the possibilities for where it might appear?

Can you state a proposition?

c. Can equal numbers appear in the same column or row? What would that imply?

d. Is it possible for the two numbers on either side of a given number to be larger than the given number? Are there other propositions like that?

e. Does the chart have to be wquare? If not, what?

f. Is the entire chart determined once you know the numbers in the top row and the left hand column?

(1) What does that imply?

(2) Are there corollaries?

3: Is this a better strategy for determining what is on the dice?

COMMENT:

It would be useful to allow the students time to play around as long as they wanted with this abstracted system, this axiomatic system, and see what propositions they came up with. It would be important, but not essential, that the propositions be "useful" in terms of the larger problem (What's on the dice?). It is crucial that students see that these axiomatic systems occur frequently as "side problems" that mathematicians work on, in the course of working out a bigger problem, and not the other way around. Usually one learns axiomatic systems, fields, groups, logic, etc., first, and then -- when applications come up, one applies them. The trouble is that nobody cares about the axiomatic system when it is presented out of context; it is poorly understood, if at all, so that no one can apply it when the time comes to do so. Furthermore, the more important business of axiomatic systems, and that is to be able to do the abstracting, is never experienced.

It is important to realize that axiomatic systems can come up anywhere in mathematics, in many different ways for different students at different times, and that it is the job of the teacher to make it clear to the students that this is happening and to help them to clarify and treat that system when it occurs.

Related Problems

Listed below are some variations of this puzzle, some of which a teacher might want to use along with, or preceeding the problem stated at the beginning. Or perhaps a teacher might want to try some of these after students have begun the one above, particularly in the case of students who cannot seem to "get anywhere" on the one above.

1. Use two coins, their faces covered with masking tape to allow numbers to be written on them, instead of the two dice. One might notice that this gives a maximum of four different sums that could occur instead of 36 as in the first problem. Experience has shown that students can solve this puzzle rather easily.
2. You might use three, or even four, coins, as indicated in no. 1 above. In that case there are 8 and 16 different sums respectively.
3. One can make a "three sided die" by requiring that every number be written on two sides of a die. Each die then has only three different numbers on its sides, all equally likely to come up. One can then play this game with two such die, or three, or more, producing a puzzle with 9, 27, or more, different sums.
4. One could combine the coins with the dice. One three sided die and a coin produce a puzzle with 6 different sums. The possibilities are endless.
5. One might repeat a number on the same die. One might repeat several numbers, but not all. One die might have, for example, the numbers 2, 2, 3, 3, 3, 7, written on it. Again, the possibilities are great for variations in this manner.
6. What happens if all six sides of both dice have 7 written on them, except for one side which has 6?
7. Supposing that you construct solid duodecahedrons out of pentagons;

Materials: Five cubes for each student.

I. Random Events. In this lesson, students should understand the conditions for randomness and realize the complexities of generating events in an order that is random. Start the class by asking each student to write down 15 digits at random.

A. In this class you should get from the students the number of times they selected each digit and the number of times each digit was selected on each draw if necessary. Say:

1. Select at random 15 digits from 0,1,2,3,4,5,6,7,8,9.

   a. How many of you selected 1 first? 2 first? 3 first? etc.

   b. How many times was 1 selected? 2 selected? 3 selected? etc.

   You may need to make a 10 x 15 table to see if the numbers were randomely selected.

2. Were the digits selected randomly?

   a. Was each digit equally likely?

   b. Was the order of occurence random?

   c. Were the outcomes of the class near the expected outcomes?

3. What would be a procedure for selecting the digits in an order that is random?

If the following is not suggested by the students, say:

B. Suppose I give each of you five cubes (Pass out the cubes.), and you label them as follows:

0,1,2,3,4,5	2,3,4,5,6,7	4,5,6,7,8,9
6,7,8,9,0,1	8,9,0,1,2,3	(Write on chalkboard.)

1. Can you use these cubes to get random numbers? How?

Let the students check their answers against the three conditions a, b and c of A-2 above. If the following procedure is not suggested, you may say:

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II. Random Sample. Organizations like the Harris and Gallop polls collect data from samples of people in this country to be used to make predictions about the entire population. For these predictions to be accurate, the samples must be representative of the entire population. The method that is used to get samples that are more likely to be representative of the entire population is known as random sampling. In this lesson students should learn some of the techniques and pitfalls of this method.

Materials: Put the following shaped blocks in a bag:

Occupation	Represented By	Percentage Of Population	Percentage of Red	Percentage of Blue
Executives	Trapezoids	10%	2%	8%
Clericla Workers	Squares	15%	9%	6%
Professionals	Circles	5%	3%	2%
Unskilled Workers	Right Triangles	25%	10%	15%
Skilled Workers	Equilateral Triangles	30%	20%	10%
Farmers	Diamonds	15%	9%	6%
		100%	53%	47%

(Put on chalkboard)

At the beginning of the class, put all of the above information on the chalkboard except the percentages of red and blue blocks. Then say:

A. On the chalkboard we have the description of the population represented by the blocks in this bag as indicated. The pieces of each shape are either red or yellow. If we use the colors to represent the way the person will vote on an issue, can you pick a sample from this population that can be used to determine the outcome of an election held by this population?

1. How would you suggest we pick this sample?

   How many blocks should be in this sample? (I suggest 20 and later 60.)

After a sample is picked, ask:

2. Is this a random sample?

   a. Was each shape equally likely? Triangles and trapezoids?

   b. Was the method of selecting a block random?

   c. Is this sample representative of the population?

   How could you find out?

3. Since each shape is not equally likely, just drawing a block is not sufficient for randomness. The students must realize that they should sort out different shapes and then draw percentage wise from each set of different shapes. Ask:

   a. According to the shape, how should the blocks in the sample be distributed for the sample to be random?

   b. Can you make sure that the blocks in the sample are distributed as they should be? How?

4. When the students decide on a method, they should use it to select a sample. Examine this sample relative to the three conditions above. Then ask:

   a. Is this a good sample? Why or why not?

   b. Is it large enough?

   c. Do you think it is representative of the population?

5. Does the issue pass or fail?

   If the students have too much difficulty finding a procedure, you should suggest sorting the blocks by shapes and drawing percentage wise from each shape.

B. After the students show some understanding of what a random sample is, they should count the two colors of blocks in the population to test their prediction based on the sample.

   They should also look at the voting of each occupational group.

Related Exercises:

1. Would students in this class be a random sample for students on this campus? Freshmen on this campus?

2. How would you select a sample that is representative of the opinions of Blacks in this state?

III. Events With Two Alternative: In this lesson, we will analyze a random system that has two possible outcomes for each simple event and some samples from that system.

Material: Each student must have 6 pennies and there must be a roll of tape that can be written on.

First, each student is to put 1 and 2 on the faces of each of four pennies.

Ask:

A. If the sum of the numbers on the upward faces is recorded after each toss, how many different sums would you get?

The students may need to toss the pennies, say, 16 times and record the results.

1. How many times did you expect each sum to turn up?

   How many different ways can you get each sum?

2. Make a distribution of the expected frequency of these sums for 16 tosses.

   What is the expected frequency of getting a sum of 4? 5? 6? etc.

B. Now, let us replace the sums with the head (H) and tail (T) notation.

(Let any combination of three heads and one tail be written as H^3T.)

1. Suppose 1 was on each head and 2 was on each tail, which sum would correspond to H^3T (3 heads and 1 tail)? T^3H? etc.

2. How many ways can you get H^3T? T^3H? etc.

3. Write down the sum of all possible outcomes using combinations of H and T.

   $(H^4 + 4H^3T + 6H^2T^2 + 4HT^3 + T^3)$

   a. Do you recognize this expression?

   b. To what is it equal?

C. What is the probability of getting 3 heads and 1 tail? 2 heads and 2 tails? etc. Substitute these probabilities for the corresponding combination of heads and tails.

1. What will the complete sum of all probabilities of possible outcomes be?

2. When can this equation be used for determining probabilities for such a sys

D. Get a sample by tossing the coins 20 times. Then analyze the data from this sample and compare the results with that of the population. Ask:

1. Was the set of 20 tosses a random sample?
2. From this data, would you predict that the population is binomially distributed?
3. How much confidence do you have in your prediction?

E. Will this work for 6 pennies ? (You may wish to use the tree diagram to help determine all the possibilities.)

Related Exercises

1. If you had ten coins with the numbers 1 and 2 on the faces, what is the probability of getting a sum of 15? 13? 20? 24? etc.
2. If the president is predicted to win 60% to 40%, what is the probability that three of the first four people picked at random will vote for him?
3. Suppose a rifleman has a probability of 0.05 of hitting the target. What is the probability that in 10 shots he never hits the target?
4. Suppose that the normal rate of infection of a certain disease is 25%. What is the probability that in a family of 6 members all will catch this disease?
5. Probability of finding an albino child is 0.1. What is the probability that in a family of 4 children at least 2 will be albino?
6. A baseball player's batting average is 0.7. Waht is the probability that he gets at least 3 hits in 5 times at bat?
7. Suppose that, in flight, airplane engines fail with probability 0.001 independently from engine to engine and that a plane makes a successful flight if at least half of its engines run. What is the probability that a 4-engine plane will fail to make a successful flight?